(Pages : 3) **T – 6370**

Reg. No. : ....

Name : ....

Second Semester M.Sc. Degree Examination, September 2024

Mathematics

MM 221 : ABSTRACT ALGEBRA

(2020–2022 Admission)

Time : 3 Hours Max. Marks : 75

PART – A

Answer any **five** questions. Each question carries **3** marks.

1. Prove that $SL(2, \mathbb{R})$ is a normal subgroup of $GL(2, \mathbb{R})$.

2. Describe the conjugacy classes of D_6.

3. Find the degree and a basis for $\mathbb{Q}\left(\sqrt{2}, \sqrt[3]{2}, \sqrt[4]{2}\right)$.

4. If E is the splitting field of $x^3 + x + 1$ over $\mathbb{Q}$, find $Gal(E/Q)$. Find all sub fields of E.

5. Find $\Phi_{21}(x)$.

6. Determine the number of elements of order 6 in $\mathbb{Z}_{12} \times \mathbb{Z}_{14} \times U(6)$.

7. Prove that a simple group of order 60 has a subgroup of order 10.

8. Find the splitting field of $x^4 + 2$ over $\mathbb{Q}$.

(5 × 3 = 15 Marks)

P.T.O.

PART – B

Answer **all** questions. Each question carries **12** marks.

9. A. (a) Suppose that G is a finite abelian group that has exactly one subgroup for each divisor of $|G|$. Show that G is cyclic. **7**

(b) Suppose that G is a non-abelian group of order p^3, where p is prime. Prove that $|Z(G)| = p$. **5**

OR

B. (a) Without doing any calculation in $Aut(\mathbb{Z}_{720})$, determine how many elements of $Aut(\mathbb{Z}_{720})$ have order 6. **5**

(b) Prove that converse of Lagrange's theorem is true for finite abelian groups. **7**

10. A. (a) Suppose that p is the smallest prime that divides $|G|$. Show that any subgroup of index p in G is normal in G. **6**

(b) Show that if G is a group of order 168 that has a normal subgroup of order 4, then G has a normal subgroup of order 28. **6**

OR

B. (a) Prove that a group of order 595 has a normal Sylow 17-subgroup. **6**

(b) Show that A_5 cannot contain a subgroup of order 20. **6**

11. A. (a) Let F be a field and $f(x) \in F[x]$. Prove that any two splitting fields of $f(x)$ over F are isomorphic. **8**

(b) Find the degree and a basis for $\mathbb{Q}\left(\sqrt{3+\sqrt{5}}\right)$ over $\mathbb{Q}\left(\sqrt{15}\right)$. **4**

OR

B. (a) Prove that an algebraic extension of an algebraic extension is algebraic. **8**

(b) Find the splitting field of $x^4 - x^2 - 2$ over $\mathbb{Z}_2$. **4**

12. A. (a) If F is a finite field, prove that $|F| = p^n$ for some $n > 0$ and some prime number p. **6**

(b) Show that any finite subgroup of the multiplicative group of a field is cyclic. **6**

OR

B. (a) Show that no finite field is algebraically closed. **6**

(b) Prove that $30°$ is a constructible angle. **6**

13. A. (a) Let F be a field of characteristic 0 and let $a \in F$. If E is the splitting field of $x^n - a$ over F, prove that the Galois group $Gal(E/F)$ is solvable. **8**

(b) Determine the Galois group of $x^3 - 2$ over $\mathbb{Q}$. **4**

OR

B. (a) Prove that $\Phi_n(x)$ is irreducible over $\mathbb{Z}$. **8**

(b) Let ω be primitive 12th root of unity over $\mathbb{Q}$. Find the minimal polynomial for ω^4 over $\mathbb{Q}$. **4**

(5 $\times$ 12 = 60 Marks)
