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Reg. No. : ....

Name : ....

First Semester M.Sc. Degree Examination, April 2024

Mathematics

MM 514 – BASIC TOPOLOGY

(2023 Admission)

Time : 3 Hours Max. Marks : 75

SECTION – A

Answer **any five** questions. Each question carries **3** marks.

1. Determine the distance from $a = (-2, 1)$ to $b = (3,4)$ in $\mathbb{R}^2$ with respect to each of the following three metrics: (a) usual metric (b) taxicab metric and (c) max metric.

2. Define the boundary of a subset *A* of a metric space *X* and show that it is always a closed set.

3. Let (X, d) be a discrete metric space and (Y, d') is any metric space. Show that any function $f : X \to Y$ is continuous.

4. Define topologically equivalent metric spaces. Show that the metric spaces $X = (0, 1)$ and $Y = (0,2)$ with the usual metric on the real line, are topologically equivalent.

5. How many different topologies are there for a set with three members?

6. Is an open mapping continuous? Justify.

7. Give example of subsets *A, B* of $\mathbb{R}^2$ to illustrate the following

   A and *B* are connected but $A \cap B$ is disconnected.

8. Give an example of a metric space having a closed and bounded subset that is not compact.

(5 × 3 = 15 Marks)

P.T.O.

SECTION – B

Answer **all** questions. Each question carries **10** marks.

9. (A) (a) Define the usual metric d for $\mathbb{R}^n$ and prove that $(\mathbb{R}^n, d)$ is a metric space.

(b) Prove that a sequence in a metric space cannot converge to more than one limit.

OR

(B) (a) Let (X, d) be a metric space and A a subset of X. Prove that a point x in X is a limit point of A if and only if there is a sequence of distinct points of A which converges to x.

(b) If A is a subset of a metric space X then prove that A is a closed set and is a subset of every closed set containing A.

10. (A) (a) Let $f : X \to Y$, where (X, d) and (Y, d') are metric spaces and $a \in X$. Then prove that f is continuous at a if and only if for each sequence $\{x_n\}$ in X converging to a, the sequence $\{f(x_n)\}$ converges to $f(a)$.

(b) Define equivalent metrics for a set X. Prove that the usual metric d and the taxicab metric d' on $\mathbb{R}^n$ are equivalent

OR

(B) (a) Let (X, d) be a complete metric space. Prove that a subspace A of a metric space X is complete if and only if it is closed.

(b) State and prove the Baire Category Theorem.

11. (A) For any subsets A, B of a topological space X, prove the following:

(a) The interior of A is the largest open set contained in A.

(b) A is open if and only if $A = \text{int}\, A$

(c) $\text{int}(A \cap B) = \text{int}\, A \cap \text{int}\, B$

(d) $\text{int}(A \cap B) = \text{int } A \cap \text{int } B$

(e) If $A \subset B$, then $\overline{A} \subset \overline{B}$

(f) $\overline{A \cup B} = \overline{A} \cup \overline{B}$

OR

(B) (a) Prove that the closed sets of a topological space X have the following properties:

(i) X and ϕ are closed sets.

(ii) The intersection of any family of closed sets is a closed set.

(iii) The union of any finite family of closed sets is a closed set.

(b) Let A be a subset of a topological space X. Prove the following for the boundary of A.

(i) A is open if and only if $bdy\ A \subset (X \setminus A)$

(ii) A is closed if and only if $bdy\ A \subset A$

(iii) A is open and closed if and only if $bdy\ A = \phi$.

12. (A) (a) Define topological property of topological spaces. Prove that separability is a topological property.

(b) Prove that a family $\mathfrak{B}$ of subsets of a set X is a basis for some topology for X if and only if both the conditions hold:

(i) The union of the members of $\mathfrak{B}$ is X.

(ii) For each members B_1, B_2 in $\mathfrak{B}$ and $x \in B_1 \cap B_2$, there is a member B_x such that $x \in B_x \subset B_1 \cap B_2$.

OR

(B) (a) Let $f : X \to Y$ be a function on the indicated topological spaces. Prove that the following statements are equivalent.

(i) f is continuous

(ii) For each closed subset C of $Y, f^{-1}(C)$ is closed in X.

(iii) For each subset A of X, $f(\overline{A}) \subseteq \overline{f(A)}$.

(b) Prove that the property of being a Hausdorff space is a topological and hereditary property.

13. (A) (a) Prove that the connected subsets of $\mathbb{R}$ are precisely intervals.

(b) Prove that every closed and bounded interval has the fixed point property.

OR

(B) (a) State and prove the Intermediate value theorem. Also, deduce the If $f : [a, b] \to R$ is continuous and $f(a)$ & $f(b)$ are of opposite signs then the equation $f(x) = 0$ has a root between *a and b.*

(b) Prove that every open, connected subset of R^n is path connected

14. (A) (a) Prove that compact subset of a Hausdorff space is closed.

(b) Prove that a continuous function $f : [a, b] \to \mathbb{R}$, whose domain is a closed and bounded interval assumes a maximum value and a minimum value.

OR

(B) (a) State and prove the Lindelof theorem.

(b) Prove that every compact space has the Bolzano-Weierstrass property.

(6 $\times$ 10 = 60 Marks)
