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Reg. No. : ....

Name : ....

First Semester M.Sc. Degree Examination, April 2024

Mathematics

MM 214 : TOPOLOGY — I

(2020 – 2022 Admission)

Time : 3 Hours Max. Marks : 75

PART – A

Answer **any five** questions. Each question carries **3** marks.

1. Define bounded set, diameter and bounded metric space.

2. Let d_1 and d_2 be two metrics for the set X and suppose that there is a positive number c such that $d_1(x,y) \leq c d_2(x,y)$ for all $x, y \in X$. Then prove that the identity function $i:(X,d_2) \to (X,d_1)$ is continuous.

3. Define nowhere dense set and give two examples.

4. For any two subsets A, B of a topological space X, prove $\text{int}(A \cap B) = \text{int} A \cap \text{int} B$.

5. Define a Hausdorff space and give an example.

6. Show that every path connected space is connected.

7. Define a connected space and give two examples.

8. Prove that a Lindel of space is compact if and only if it is countably compact.

(5 × 3 = 15 Marks)

P.T.O.

PART – B

Answer **all** questions. Each question carries **12** marks.

9. A. (a) State and prove the Cauchy-Schwarz inequality. **6**

(b) Let (X, d) be a metric space and A, a subset of X. Prove that a point x in X is a limit point of A if and only if there is a sequence of distinct points of A which converges to X. **6**

OR

B. (a) Prove that if A is a subset of a metric space X, then $\overline{A}$ is a closed set and is a subset of every closed set containing A. **4**

(b) Prove that singleton sets are open in a discrete metric space. **4**

(c) Define the boundary of a subset of metric space X and give two examples. **4**

10. A. Let (X, d) be a metric space. Show that there is a complete metric space (Y, d') and an isometric embedding $e: X \to Y$ for which $e(X)$ is a dense subspace of Y. Also prove that the space (Y, d') is unique up to metric equivalence.

OR

B. (a) Prove that the following statements are equivalent for a function f from metric space (X, d) to metric space (Y, d') :

(i) f is continuous.

(ii) For each sequence $\{x_n\}_{n=1}^{\infty}$ converging to a point a in X, the sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(a)$.

(iii) For each open set O in Y, $f^{-1}(O)$ is open in X.

(iv) For each closed set C in Y, $f^{-1}(C)$ is closed in X. **6**

(b) State and Prove Baire Category theorem. **6**

11. A. (a) Show that a Hilbert space H is second countable. **6**

(b) Prove : Separability is a topological property. **6**

OR

B. Let $f : X \rightarrow Y$ be a function on the indicated topological spaces. Prove that the following statements are equivalent.

(a) f is continuous.

(b) For each closed subset C *of* $Y, f^{-1}(C)$ is closed in X.

(c) For each subset A of X, $f(\overline{A}) \subset \overline{F(A)}$.

(d) There is a basis $\mathcal{B}$ for the topology of Y such that $f^{-1}(B)$ is open in X for each basic open set B in $\mathcal{B}$

(e) There is a subbasis S for the topology of Y such that $f^{-1}(S)$ is open in X for each subbasic open set S in S.

12. A. Show that the following statements are equivalent for a topological space X.

(a) X is disconnected

(b) X is the union of two disjoint, non-empty closed sets.

(c) X is the union of two separated sets.

(d) There is a continuous function from X onto a discrete two-point space $\{a, b\}$.

(e) X has a proper subset A which is both open and closed.

(f) X has a proper subset A such that $\overline{A} \cap \overline{(X \setminus A)} = \phi$.

OR

B. (a) Prove that the connected subsets of $\mathbb{R}$ are precisely the intervals. **6**

(b) Show that every open, connected subset of $\mathbb{R}^n$ is path connected. **6**

13. A. (a) Prove : Each closed subset of a compact space is compact **4**

(b) Let X be a compact space, Y a space and $f: X \to Y$ a continuous function from X onto Y. Then prove that Y is compact. **4**

(c) State and prove the Lindelof Theorem. **4**

OR

B. (a) Let (X, d) be a compact metric space, (Y, d') a metric space, and $f: X \to Y$ a continuous function. Then prove that f is uniformly continuous. **6**

(b) Show that the Cantor set is a compact, perfect, totally disconnected metric space **6**

(5 $\times$ 12 = 60 Marks)