

Reg. No. : .....

Name : .....

Third Semester M.Sc. Degree Examination, February 2024

Mathematics

MM 231 : COMPLEX ANALYSIS

(2020 Admission onwards)

Time : 3 Hours

Max. Marks : 75

PART – A

Answer **any five** questions. Each question carries **3** marks.

1. Find the radius of convergence of  $\sum_{n=0}^{\infty} k^n z^n$ ,  $k$  an integer.
2. Give the power series expansion of  $\log z$  about  $z=i$ .
3. Let  $f$  be analytic on  $B(0;1)$  and suppose  $|f(z)| \leq 1$  for  $|z| < 1$ . Show that  $|f'(0)| \leq 1$ .
4. Determine the nature of  $f(z) = \frac{\log(z+1)}{z^2}$ .
5. Evaluate the cross ratio of  $(2, 1-i, 1, 1+i)$ .
6. Evaluate  $\int_{\gamma} \frac{\sin z}{z^3}$ ,  $\gamma(t) = e^{it}$ ,  $0 \leq t \leq 2\pi$ .
7. Suppose that  $f:G \rightarrow \mathbb{C}$  is analytic and one to one. Show that  $f'(z) \neq 0$  for any  $z \in G$ .
8. If  $S$  is a Mobius transformation, prove that  $S$  is the composition of translations, dilations, and the inversion.

(5 × 3 = 15 Marks)

P.T.O.



PART – B

Answer **all** questions. Each question carries **12** marks.

9. (A) Let  $f$  be analytic in  $B(a;R)$ . Prove that  $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$  for  $|z-a| < R$  where  $a_n = \frac{1}{n!} f^{(n)}(a)$  and this series has radius of convergence  $\geq R$ .

OR

- (B) If  $\gamma: [a, b] \rightarrow \mathbb{C}$  is piecewise smooth, prove that  $\gamma$  is bounded variation and  $V(\gamma) = \int_a^b |\gamma'(t)| dt$ .
10. (A) Let  $G$  be a connected open set and let  $f: G \rightarrow \mathbb{C}$  be an analytic function. Prove that following are equivalent:
- (a)  $f \equiv 0$
  - (b) there is a point  $a$  in  $G$  such that  $f^{(n)}(a) = 0$  for each  $n \geq 0$
  - (c)  $\{z \in G: f(z) = 0\}$  has a limit point in  $G$ .

OR

- (B) (a) If  $\gamma: [0, 1] \rightarrow \mathbb{C}$  is a closed rectifiable curve and  $a \notin \{\gamma\}$ , Show that  $\frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z-a}$  is an integer. **6**
- (b) Evaluate  $\int_{\gamma} \frac{\sin z}{z} dz$  where  $\gamma(t) = e^{it}, 0 \leq t \leq 2\pi$ .
11. (A) (a) State and prove Morera's theorem. **8**
- (b) Let  $G$  be simply connected and let  $f: G \rightarrow \mathbb{C}$  be an analytic function such that  $f(z) \neq 0$  for any  $z \in G$ . Prove that there is an analytic function  $g: G \rightarrow \mathbb{C}$  such that  $f(z) = \exp g(z)$ . **4**

OR

- (B) (a) State and prove the independence of path theorem. **6**
- (b) Suppose  $f$  is analytic in  $B(a;R)$  and let  $\alpha = f(a)$ . If  $f(z) - \alpha$  has a zero of order  $m$  at  $z = a$ , Show that there is an  $\varepsilon > 0$  and  $\delta > 0$  such that for  $|\zeta - \alpha| < \delta$ , the equation  $f(z) = \zeta$  has exactly  $m$  simple roots in  $B(a; \varepsilon)$ . **6**



12. (A) Prove that  $\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$ .

OR

(B) (a) State and prove Argument principle. 7

(b) Give the partial fraction expansion of  $r(z) = \frac{z^2 + 1}{(z^2 + z + 1)(z - 1)^2}$ . 5

13. (A) (a) Let  $z_1, z_2, z_3, z_4$  be four distinct points in  $\mathbb{C}_{\infty}$ . Prove that  $(z_1, z_2, z_3, z_4)$  is a real number if and only if all four points lie on a circle. 6

(b) State and prove Schwarz's lemma. 6

OR

(B) (a) With usual notation, prove that if  $|a| < 1$ , then  $\phi_a$  is a one-one map of  $D = \{z: |z| < 1\}$  onto itself; the inverse of  $\phi_a$  is  $\phi_{-a}$ . 6

(b) If  $f: G \rightarrow \mathbb{C}$  is analytic, Prove that  $f$  preserves angles at each point  $z_0$  of  $G$  where  $f'(z_0) \neq 0$ . 6

**(5 × 12 = 60 Marks)**

