

Reg. No. :

Name :

Fourth Semester M.Sc. Degree Examination, July 2024

Mathematics

Elective II

MM 244.3 – ADVANCED COMPLEX ANALYSIS

(2020 Admission Onwards)

Time : 3 Hours

Max. Marks : 75

PART – A

Answer any **five** questions. Each question carries **3** marks.

1. Let G be an open set in \mathbb{C} and (Ω, d) be a complete metric space. Define a normal subset of the set of all continuous functions from G to Ω .
2. Can the space of complex numbers be equivalent to any bounded region? Justify.
3. Prove that if $|z| < \frac{1}{2}$ then $\frac{1}{2}|z| \leq |\log(1+z)| \leq \frac{3}{2}|z|$.
4. Find a factorization of $\cosh z$.
5. Define the Riemann zeta function. With usual notations, prove that for $\operatorname{Re} z > 1$,
$$\zeta(z)\Gamma(z) = \int_0^{\infty} (e^t - 1)^{-1} t^{z-1} dt.$$

P.T.O.



6. Let f be an analytic function on G . Define the germ of f at a , $[f]_a$. Prove that $(g, D) \in [f]_a$ if and only if $(f, G) \in [g]_a$.
7. Let G be a region and suppose that u is a continuous real valued function on G with the mean value property. Prove that if there is a point a in G such that $u(a) \geq u(z)$ for all z in G , then u is a constant function.
8. Prove that if f is an entire function of order λ then f' also has order λ .

(5 × 3 = 15 Marks)

PART – B

Answer **all** questions. Each question carries **12** marks.

9. (A) (a) Let G be an open set in \mathbb{C} and (Ω, d) be a complete metric space. Prove that the space of all continuous functions from G to Ω , $\mathbb{C}(G, \Omega)$ is a complete metric space.

(b) State and prove Montel's theorem. **6+6**

OR

(B) State and prove the Arzela-Ascoli theorem. **12**

10. (A) (a) Prove that $\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$.

(b) State and prove the Bohr-Mollerup theorem. **5+7**

OR

(B) Let G be a region and let $\{a_j\}$ be a sequence of distinct points in G with no limit points in G . Let $\{m_j\}$ be a sequence of integers. Prove that there is an analytic function f defined on G whose only zeros are at the points a_j . Also prove that a_j is a zero of f of multiplicity m_j . **12**

11. (A) (a) Derive Riemann's functional equation

$$\zeta(z) = 2(2\pi)^{z-1} \Gamma(1-z) \zeta(1-z) \sin\left(\frac{1}{2}\pi z\right) \text{ for } -1 < \text{Re } z < 0.$$



- (b) Let K be a compact subset of the region G . Prove that there are straight line segments $\gamma_1, \dots, \gamma_n$ in $G - K$ such that for every function f in $H(G)$, $f(z) = \sum_{k=1}^n \frac{1}{2\pi i} \int_{\gamma_k} \frac{f(w)}{w-z} dw$ for all z in K . **6+6**

OR

- (B) State and prove the Mittag-Leffler's theorem.

12. (A) State and prove Schwarz reflection principle.

OR

- (B) (a) Let $\gamma: [0,1] \rightarrow \mathbb{C}$ be a path from a to b and let $\{(f_t, D_t): 0 \leq t \leq 1\}$ and $\{(g_t, B_t): 0 \leq t \leq 1\}$ be analytic continuations along γ such that $[f_0]_a = [g_0]_a$. Then prove that $[f_1]_b = [g_1]_b$.

- (b) State and prove Monodromy theorem. **6+6**

13. (A) (a) Let G be a region and suppose that u and v are two continuous real valued functions on G with mean value property. Prove that if there is a point a in the extended boundary $\partial_\infty G$, $\lim_{z \rightarrow a} \sup u(z) \leq \lim_{z \rightarrow a} \inf v(z)$ then either $u(z) < v(z)$ for all z in G or $u = v$.

- (b) Prove that if $u: G \rightarrow \mathbb{R}$ is a continuous function which has the Mean value property that u is harmonic. **7+5**

OR

- (B) (a) Derive Jensen's formula.

- (b) State and prove the Hadamard factorization theorem. **5+7**

(5 × 12 = 60 Marks)

