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Reg. No. : ....

Name : ....

Fourth Semester M.Sc. Degree Examination, July 2024

Mathematics

MM 241 : ANALYTIC NUMBER THEORY

(2020 Admission Onwards)

Time : 3 Hours Max. Marks : 75

PART – A

Answer any **five** questions. Each question carries **3** marks.

1. Define multiplicative arithmetic functions and completely multiplicative arithmetic functions. Give examples for both.

2. Prove or disprove with reason. The Dirichlet inverse of a multiplicative function is also multiplicative.

3. Define the group of reduced residue classes modulo k, where k is some fixed integer. Find the order of this group.

4. Prove that there are infinitely many primes of the form $4n+1$.

5. Determine those odd primes p for which 3 is a quadratic residue and for those which it is nonresidue.

6. State and prove reciprocity law for Jacobi symbols.

7. Prove that m is prime if and only if $\exp_m(a) = m-1$ for some a.

8. Prove that 3 is a primitive root mod p if p is a prime of the form $2^n+1, n>1$.

(5 × 3 = 15 Marks)

P.T.O.

PART – B

Answer **all** questions. Each question carries **12** marks.

9. (A) Prove that the set of all arithmetical functions f with $f(1) \neq 0$ forms an abelian group with respect to the Dirichlet convolution. **12**

OR

(B) (i) Derive the relation between Möbius function and Euler Totient function. **6**

(ii) Enlist the five properties of Euler's Totient and prove. **6**

10. (A) Prove that a finite abelian group G of order n has exactly n distinct characters. **12**

OR

(B) (i) For any real-valued nonprincipal character χ mod k, let $A(n) = \sum_{n \leq x} \frac{A(n)}{\sqrt{n}}$, then prove the following :

(1) $B(x) \to \infty$ as $x \to \infty$

(2) $B(x) = 2\sqrt{x}\ L(1, \chi) + O(1)$ for all $x \geq 1$, therefore $L(1, \chi) \neq 0$. **8**

(ii) Define multiplication of characters in such a way that the set of characters of a finite abelian group G becomes a group under that multiplication. Show that the set satisfies the group postulates. **4**

11. (A) (i) Define Asymptotic equality of functions with example. **4**

(ii) If $\chi \neq \chi_1$ and $L(1, \chi = 0)$, prove $L'(1, \chi) \sum_{n \leq x} \frac{\mu(n)\chi(n)}{n} = \log x + O(1)$. **8**

OR

(B) For $x > 1$ and $\chi \neq \chi_1$, prove that $\sum_{p \leq x} \frac{\chi(p) \log p}{p} = -L'(1, \chi) \sum_{n \leq x} \frac{\mu(n)\chi(n)}{n} + O(1)$.

12

12. (A) (i) State and prove Euler's criterion of Legendre's symbols. **4**

(ii) Show that the Diophantine equation $y^2 = x^3 + k$ has no solution if k is of the form $k = (4n+1)^3 - 4m^2$, where m and n are integers such that no prime $p \equiv -1 \pmod 4$ divides m. **8**

OR

(B) State and Prove Gauss Lemma. **12**

13. (A) Let p be an odd prime and let d be any positive divisor of $p-1$. Prove that in every reduced residue system $\bmod p$, there are exactly $\phi(d)$ numbers 'a' such that $\exp_p(a) = d$. **12**

OR

(B) (i) Let p be an odd prime and $\alpha \geq 1$, prove that there exists odd primitive roots g modulo p^α. Show also that each such g is also a primitive root modulo $2p^\alpha$. **8**

(ii) Let g be a primitive root modulo p, which integral powers of g are quadratic residues $\bmod p$ and which integral powers are quadratic nonresidues $\bmod p$. Prove. **4**

(5 × 12 = 60 Marks)
