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Reg. No. : ....

Name : ....

Fourth Semester M.Sc. Degree Examination, July 2024

Mathematics

MM 242 : FUNCTIONAL ANALYSIS – II

(2020 Admission Onwards)

Time : 3 Hours Max. Marks : 75

SECTION – A

Answer any **five** questions. Each question carries **3** marks.

1. Show that 0 can be spectral value of a compact operator without being its eigenvalue.
2. State and prove the Schwarz inequality.
3. Show that an inner product space is uniformly convex in the norm.
4. State and prove the Polarization identity.
5. Show that the projection theorem does not hold for an incomplete inner product space.
6. Show that a normal operator need not be neither unitary nor self-adjoint.
7. Let $H \neq \{0\}$ and let $A \in BL(H)$ be self-adjoint. Show that $\|A\| = \max\{|m_A|\} = \sup\{|k| : k \in \sigma(A)\}$.
8. Show that if $K = R$, then a normal operator on a finite dimensional space may not have any eigenvectors.

(5 × 3 = 15 Marks)

P.T.O.

SECTION – B

Answer **all** questions. Each question carries **12** marks.

9\. (a) (i) Let X be a normed space and $A \in CL(X)$. Show that every nonzero spectral value of A is an eigenvalue of A. **8**

(ii) Let X be an infinite dimensional normed space and $A \in CL(X)$. Show that $0 \in \sigma_a(A)$. **4**

OR

(b) (i) Let X be a normed space and $A \in CL(X)$. Show that the eigen-spectrum and the spectrum of A are countable sets and have 0 as the only possible limit point. In particular, if $\{k_1, k_2, ...\}$ is an infinite set of eigen values of A then show that $k_n \to 0$ as $n \to \infty$. **6**

(ii) Let X be normed space and $A \in CL(X)$. Show that every eigenspace of A corresponding to a nonzero eigenvalue of A is finite dimensional. **6**

10\. (a) (i) State and prove the Bessel's inequality. **6**

(ii) Let H be a nonzero Hilbert space over K. Show that the following conditions are equivalent:

(1) H has a countable orthonormal basis

(2) H is linearly isothermic to K'' for some n or to l^2

(3) H is separable. **6**

OR

(b) (i) State and prove Gramm–Schmidt orthonormalization process. **8**

(ii) Let $\{u_\alpha\}$ be an orthonormal set in an inner product space X and $x \in X$. Let $E_x = \{u_\alpha : \langle x, u_\alpha \rangle \neq 0\}$. Then show that E_x is a countable set, say $E_x = \{u_{1,}, u_2, ...\}$ and if E_x is denumerable the n$\langle x, u_n \rangle \to 0$ as $n \to \infty$. **4**

11. (a) Let X be an inner product space.

(i) Let $E \subset X$ and $x \in \overline{E}$. Show that there exists a best approximation from E to x if and only if $X \in E$. **3**

(ii) If $E \subset X$ is convex, then show that there exits at most one best approximation from E to any $x \in X$. **3**

(iii) Let F be a subspace of X and $x \in X$. Show that $y \in F$ is a best approximation from F to x if and only if $x - y \perp F$ and in that case dist $(x, F) = \langle x, x - y \rangle^{1/2}$. **6**

OR

(b) (i) State and prove the Projection theorem. **6**

(ii) Let X be an inner product space and $f \in X'$. Let $\{u_1, u_2, ...\}$ be an orthonormal set in X. Show that $\sum_n |f(u_n)|^2 \le \|f\|^2$. **3**

(iii) Let X be an inner product space and $f \in X'$. Let $\{u_\alpha\}$ be an orthonormal set in X and $E_f = \{u_\alpha : f(u_\alpha) \neq 0\}$. Then show that E_f is a countable set, say $\{u_1, u_2, ...\}$ and if E_f is denumberable, then $f(u_n) \to 0$ as $n \to \infty$. **3**

12. (a) (i) Let H be a Hilbert space and $A \in BL(H)$. Show that there is a unique $B \in BL(H)$ such that for all $x, y \in H$, $\langle A(x), y \rangle = \langle x, B(y) \rangle$. **4**

(ii) Let H be a Hilbert space. Consider $A, B \in BL(H)$ and $\in K$. Show that $(A + B)^* = A^* + B^*, (kA) = \bar{k}A^*, (AB) = B^* A^*$, $(A^*)^* = A$. Further show that A is invertible if and only if A^* is invertible and in that case $(A^*)^{-1} = (A^{-1})^*$. **4**

(iii) Let H be a Hilbert space and $A \in BL(H)$. Show that $\|A^*\| = \|A\|$ and $\|A^*A\| = \|A\|^2 = \|AA^*\|$. **4**

OR

(b) (i) Let H be a Hilbert space and $A \in BL(H)$ be self-adjoint. Show that $\|A\| = \sup\{|\langle A(x), x\rangle| : x \in H, \|x\| \le 1\}$. Show also that $A = 0$ if and only if $\langle A(x), x\rangle = 0$ for all $x \in H$. **6**

(ii) Let H be a Hilbert space and $A \in BL(H)$. Show that A is unitary if and only if $\|A(x)\| = \|x\|$ for all $x \in H$ and A is surjective. In that case show that $\|A^{-1}(x)\| = \|x\|$ for all $x \in H$ and $\|A\| = 1 = \|A^{-1}\|$. **6**

13. (a) Let $A \in BL(H)$.

(i) If $R(A)$ is finite dimensional, show that A is compact. **6**

(ii) If each A_n is compact and $\|A_n - A\| \to 0$, then show that A is compact. **6**

OR

(b) Let A be a compact operator on $H \neq \{0\}$.

(i) Show that every nonzero approximate eigenvalue of A is, in fact, an eigenvalue of A and the corresponding eigenspace is finite dimensional. **6**

(ii) If A is self-adjoint, then show that $\|A\|$ or $-\|A\|$ is an eigenvalue of A. **6**

(5 × 12 = 60 Marks)