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Reg. No. : ....

Name : ....

Second Semester M.Sc. Degree Examination, September 2024

Mathematics

MM 521 – ABSTRACT ALGEBRA

(2023 Admission)

Time : 3 Hours Max. Marks : 75

PART – A

Answer any **five** questions. Each question carries **3** marks.

1. How many elements of order 9 does $Z_3 \oplus Z_9$ have?

2. What is the smallest positive integer *n* such that there are two nonisomorphic groups of order *n?* Name the two groups.

3. What can you say about the number of elements of order 7 in a group of order $168 = 8 \cdot 3 \cdot 7$?

4. Prove that A_6 has no subgroup of order 120.

5. Find the splitting field of $x^3 - 1$ over *Q.*

6. Let a be a zero of $x^3 + x^2 + 1$ in some extension field of Z_2. Find the multiplicative inverse of a + 1 in $Z_2[\alpha]$.

7. Let *E* be the algebraic closure of *F.* Show that every polynomial in $F[x]$ splits in E.

8. Determine the group of field automorphisms of $GF[4]$.

(5 × 3 = 15 Marks)

P.T.O.

PART – B

Answer **all** questions. Each question carries **10** marks.

9. A. (a) Prove that $G_1 \oplus G_2$ is isomorphic to $G_2 \oplus G_1$. State the general case. **5**

(b) Let G and H be finite cyclic groups. Then $G \oplus H$ is cyclic if and only if $|G|$ and $|H|$ are relatively prime. **5**

OR

B. (a) Let G be a group and let H be a normal subgroup of G. Show that the set $G/H = \{aH \mid a \in G\}$ is a group under the operation $(aH)(bH) = abH$. **5**

(b) Determine all subgroups of R^* (nonzero reals under multiplication) of index 2. **5**

10. A. (a) Determine all homomorphisms from Z_{12} to Z_{30}. **5**

(b) Suppose that G is a finite Abelian group that has exactly one subgroup for each divisor of $|G|$ Show that G is cyclic. **5**

OR

B. (a) Show that every group of order 35 is cyclic. **7**

(b) Prove that any Abelian group of order 45 has an element of order 15. **3**

11. A. (a) Let G be a nontrivial finite group whose order is *a* power of a prime p. Then show that $Z(G)$ has more than one element. **5**

(b) Show that Z_2 is the only group that has exactly two conjugacy classes. **5**

OR

B. (a) Let H be a subgroup of a group G. Prove that the number of conjugates of H in G is $|G : N(H)|$. **5**

(b) Prove that there is no simple group of order $528 = 2^4 \cdot 3 \cdot 11$. **5**

12. A. (a) Let F be a field and let $f(x)$ be a nonconstant polynomial in $F[x]$. Show that there is an extension field E of F in which $f(x)$ has a zero. **6**

(b) Show that every finite field is perfect. **4**

OR

B. (a) If K is an algebraic extension of E and E is an algebraic extension of F, then show that K is an algebraic extension of F. **5**

(b) Prove that $Q\left(\sqrt{2}, \sqrt[3]{2}, \sqrt[4]{2}, \ldots\right)$ is an algebraic extension of Q but not a finite extension of Q. **5**

13. A. (a) Show that for each prime p and each positive integer n, there is, up to isomorphism, a unique finite field of order p^n. **6**

(b) Prove that the degree of any irreducible factor of $x^8 - x$ over Z_2 is 1 or 3. **4**

OR

B. (a) Let α be a zero of $f(x) = x^2 + 2x + 2$ in some extension field of Z_3. Find the other zero of $f(x)$ in $Z_3[\alpha]$. **5**

(b) If a and b are constructible, show that ab is constructible. **5**

14. A. (a) Show that a factor group of a solvable group is solvable. **5**

(b) Let N be a normal subgroup of a group G. If both N and G/N are solvable, then show that G is solvable. **5**

OR

B. (a) For every positive integer n, prove that $x^n - 1 = \prod_{d|n} \Phi_d(x)$, where the product runs over all positive divisors d of n. **5**

(b) Prove that $\Phi_{2n}(x) = \Phi_n(-x)$ for all odd integers $n > 1$. **5**

(6 $\times$ 10 = 60 Marks)