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Reg. No. : ....

Name : ....

Second Semester M.Sc. Degree Examination, September 2024

Mathematics

MM 522 : MEASURE THEORY

(2023 Admission)

Time : 3 Hours Max. Marks : 75

SECTION – A

Answer any **five** questions. Each question carries **3** marks.

1. Let X be a σ-algebra of subsets of a set X. If μ is a measure on X and A is a fixed set in X, then prove that the function λ, defined for $E \in X$ by $\lambda(E) = \mu(A \cap E)$, is a measure on X.

2. Let X be an uncountable set and X be the collection of subsets which are either countable or have countable complements. Prove that X is a σ-algebra.

3. Define a simple function. Set ϕ be a simple function in $M^{+}(X, X)$ and $C \geq 0$, then prove that $\int C\phi \, d\mu = C\int \phi d\mu$.

4. Suppose that f belongs to M^{+}. Then prove that $f(x) = 0$, μ–almost everywhere on X if and only if $\int f \, d\mu = 0$.

5. Let $f_n = \chi_{[n, n+1]}$. Show that the sequence (f_n) converges everywhere to the 0-function, but it does not converge in measure.

6. If E is a countable subset of $\mathbb{R}$, then prove that it has Lebesgue measure zero.

P.T.O.

7. If $[E_k]$ is an increasing sequence of measurable sets, then prove that $m\left(\bigcup_{k=1}^{\infty} E_k\right) = \lim_{k\to\infty} m(E_k)$.

8. Prove that the outer measure of the empty set is zero.

(5 × 3 = 15 Marks)

SECTION – B

Answer **all** questions. Each question carries **10** marks.

9. (A) (a) Let f and g be measurable real-valued functions and let C be a real number. Then prove that the functions $cf, f^2, f+g, fg, |f|$ are also measurable. **5**

(b) Let (f_n) be a sequence in $M(x, X)$ and define the functions

$f(x) = \inf f_n(x)$, $F(x) = \sup f_n(x)$,

$f^*(x) = \lim \inf f_n(x)$, $F^*(x) = \lim \sup f_n(x)$.

Then prove that f, F, f^* and F^* belongs to $M(x, X)$. **5**

OR

(B) If f is a non negative function in $M(x, X)$, then prove that there exists a sequence (ϕ_n) in $M(x, X)$ such that

(a) $0 \le \phi_n(x) \le \phi_{n+1}(x)$ for $x \in X$, $n \in \mathbb{N}$

(b) $f(x) = \lim \phi_n(x)$ for each, $x \in X$

(c) Each ϕ_n has only a finite number of real values. **10**

10. (A) (a) State and prove monotone convergence theorem. **5**

(b) If f is measurable, g is integrable, and $|f| \leq |g|$, then prove that fg is integrable, and $\int |f| \, d\mu \leq \int |g| \, d\mu$. **5**

OR

(B) (a) State and prove Fatou's lemma. **5**

(b) If f belongs to M^+ and if λ is defined on X by $\lambda(E) = \int_E f \, d\mu$, then prove that λ is a measure. **5**

11. (A) (a) State and prove holders inequality. **5**

(b) Suppose that $\mu(X) < \infty$ and that (f_n) is a sequence in L_P which converges uniformly on X to f. Then prove that f belongs to L_p and the sequence (f_n) converges in L_p to f. **5**

OR

(B) If $1 \leq p < \infty$, then prove that the space L_p is a complete normed linear space under the norm $\|f\|_p = \left\{ \int |f|^p \, d\mu \right\}^{1/p}$. **10**

12. (A) (a) If λ is a charge on X, then prove that there exist sets P and N in X with $X = P \cup N$, $P \cap N = \phi$ and such that P is positive and N is negative with respect to λ. **5**

(b) If P_1, N_1 and P_2, N_2 are Hahn decompositions for λ, and E belongs to X, then prove that

$$\lambda(E \cap P_1) = \lambda(E \cap P_2)$$
$$\lambda(E \cap N_1) = \lambda(E \cap N_2).$$ **5**

OR

(B) (a) State and prove Lebesgue decomposition theorem. **5**

(b) State and prove Hahn Extension Theorem. **5**

13. (A) (a) If (E_k) is a sequence of subsets of $\mathbb{R}^p$, then prove that $m^*\left(\bigcup_{k=1}^{\infty} E_k\right) \le \sum_{k=1}^{\infty} m^*(E_k)$. **5**

(b) If I is any cell in $\mathbb{R}^p$, then prove that $m^*(I) = \ell(I)$. **5**

OR

(B) (a) If $E \subseteq \mathbb{R}^p$ and $x \in \mathbb{R}^p$, then prove that $m^*(X \oplus E) = m^*(E)$. **5**

(b) Let A and B be disjoint subsets of $\mathbb{R}^p$ with $d(A, B) > 0$. Then prove that $m^*(A \cup B) = m^*(A) + m^*(B)$. **5**

14. (A) (a) Prove that every open and every closed subset of $\mathbb{R}^p$ is Lebesque measurable. **5**

(b) If $E \subseteq \mathbb{R}^p$ is Lebesgue measurable and $x \in \mathbb{R}^p$, then prove that $X \oplus E$ is Lebesgue measurable and $m(X \oplus E) = m(E)$. **5**

OR

(B) (a) If $Z \subseteq \mathbb{R}^p$ is a null set, then prove that Z is a Lebesgue measurable set and $m(z) = 0$. Also prove that any subset of Z is Lebesgue measurable and a null set. **5**

(b) If (F_k) is a decreasing sequence of Lebesgue measurable sets and if $m(F_1) < \infty$, then prove that $m\left(\bigcap_{k=1}^{\infty} F_k\right) = \lim_{k \to \infty} m\,(F_k)$. **5**

(6 $\times$ 10 = 60 Marks)