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Reg. No. : ....

Name : ....

Second Semester M.Sc. Degree Examination, September 2024

Mathematics

MM 224 : PARTIAL DIFFERENTIAL EQUATIONS AND INTEGRAL EQUATIONS

(2020 – 2022 Admission)

Time : 3 Hours Max. Marks : 75

PART – A

Answer **any five** questions. **Each** question carries **3** marks.

1. Find α and β such that the equation $u_{xx} + u_{yy} = 5e^{x-2y}$ has a solution of the form $u(x,y) = e^{\alpha x + \beta y}$.

2. Solve the equation $u_x + u_y = 1$ with the initial condition $u(x,0) = f(x)$.

3. Let $u(x, t)$ be the solution of the Cauchy problem.

$$u_{tt} - 9u_{xx} = 0, -\infty < x < \infty, t > 0$$

$$u(x, 0) = \begin{cases} 1, & |x| \le 2 \\ 0, & |x| > 2 \end{cases}$$

$$u_t(x, 0) = \begin{cases} 1, & |x| \le 2 \\ 0, & |x| > 2 \end{cases}$$

Find $u(0, \frac{1}{6})$.

P.T.O.

4. Prove that the Cauchy problem $\begin{array}{ll} u_{tt} - c^2 u_{xx} = F(x, t), & -\infty < x < \infty, t > 0 \\ u(x, 0) = f(x), u_t(x, 0) = g(x), & -\infty < x < \infty \end{array}$ admits at most one solution.

5. Prove that the solution to the Dirichlet problem is unique.

6. Let $u(x,y)$ be a non-constant harmonic function in the disk $x^2 + y^2 < R^2$. Define for each $0 < r < R$ $M(r) = \max_{x^2+y^2=r^2} u(x,y)$. Prove that $M(r)$ is a monotone increasing function in the interval $(0, R)$.

7. Find the iterated kernel $K_2(x,\xi)$ associated with $K(x,\xi) = x - \xi$ in the interval $(0, 1)$.

8. Define isoperimetric problem. Give an example.

(5 × 3 = 15 Marks)

PART – B

Answer **all** questions. **Each** question carries **12** marks.

9. A. Consider the Cauchy problem $F(x,y,u,u_x,u_y) = 0$, $x = x_0(s)$, $y = y_0(s)$, $u = u_0(s)$. Assume that the generalized transversality conditions.

$$F(P_0) = 0,\ u_0'(s_0) = p_0(s_0)\, x_0'(s_0) + q_0(s_0)\, y_0'(s_0)\, x_0'(s_0)$$

$$F_q(P_0) - y_0'(s_0)\, F_p(P_0) \neq 0$$

hold at P_0. Prove that there exists $\in > 0$ and a unique solution $(x(t,s), y(t,s), u(t,s), p(t,s), q(t,s))$ for the Cauchy problem which is defined for $|s - s_0| + |t| < \varepsilon$. **12**

OR

B. (a) Use the Lagrange method to find a function $u(x,y)$ that solves the problem. **6**

$$-y u_x + x u_y = 0$$
$$u(x, 0) = \sin x, x > 0$$

(b) Show that the Cauchy problem $u_x + u_y = 1, u(x,x) = x$ has infinitely many solutions. **6**

10. A. Consider the equation $xu_{xx} - yu_{yy} + \frac{1}{2}(u_x - u_y) = 0$.

(a) Find the domain where the equation is elliptic, and the domain where it is hyperbolic. **3**

(b) For each of the above two domains, find the corresponding canonical transformation. **9**

OR

B. (a) Solve the problem : **6**

$$\begin{aligned} &u_{tt} - u_{xx} = t^7, && -\infty < x < \infty, t > 0 \\ &u(x, 0) = 2x + \sin x, && -\infty < x < \infty \\ &u_t(x, 0) = 0, && -\infty < x < \infty \end{aligned}$$

(b) Find the general solution of the equation. **6**

$$u_{xx} - 2\sin x\, u_{xy} - \cos^2 x\, u_{yy} - \cos x\, u_y = 0.$$

11. A. (a) Solve the problem : **6**

$$\begin{aligned} &u_{tt} - 4u_{xx} = 0, && 0 < x < 1, t > 0 \\ &u_x(0, t) = u_x(1, t) = 0, && t \geq 0 \\ &u(x, 0) = f(x) = \cos^2 \pi x, && 0 \leq x \leq 1 \\ &u_t(x, 0) = g(x) = \sin^2 \pi x \cdot \cos \pi x, && 0 \leq x \leq 1 \end{aligned}$$

(b) State and prove the strong maximum principle of harmonic functions. **6**

OR

B. (a) Solve the equation $u_t = 17u_{xx}, 0 < x < \pi, t > 0$ with the boundary conditions $u(0, t) = u(\pi, t) = 0, t \geq 0$ and the initial conditions $u(x, 0) = \begin{cases} 0, & 0 \leq x \leq \frac{\pi}{2} \\ 2, & \frac{\pi}{2} < x \leq \pi \end{cases}$. **6**

(b) Solve the Laplace equation in the square $0 < x, y < \pi$, subject to the Dirichlet conditions. **6**

$$u(x, 0) = 1984, u(x, \pi) = u(0, y) = u(\pi, y) = 0.$$

12. A. (a) Explain the relation between an initial value problem consisting of a second order linear non-homogeneous equation and a Volterra integral equation. **6**

(b) By the method of successive approximations, solve **6**

$$y(x) = 1 + \lambda \int_0^1 (1 - 3x\xi) y(\xi) d\xi .$$

OR

B. Determine the characteristics values of λ and the corresponding characteristics functions for the equation $y(x) = \lambda \int_0^{2\pi} \sin(x + \xi) y(\xi) d\xi$. **12**

13. A. (a) Find the external for the integral $I = \int_{x_1}^{x_2} f(x, y, y')\, dx$, if the integrand is $y^2 - (y')^2$. **6**

(b) Show that the geodesics on a sphere are arcs of great circles. **6**

OR

B. (a) Find the shortest curve joining two points (x_1, y_1) and (x_2, y_2). **6**

(b) A uniform flexible chain of given length hangs between two points. Find its shape if it hangs in such a way as to minimize its potential energy. **6**

(5 × 12 = 60 Marks)