(Pages : 6) **M – 2613**

Reg. No. : ....

Name : ....

Second Semester B.Sc. Degree Examination, December 2021

Career Related First Degree Programme Under CBCSS

Mathematics

Complementary Course II for Physics and Computer Applications

MM 1231.6 : MATHEMATICS – II PARTIAL DIFFERENTIATION, VECTOR DIFFERENTIATION, COMPLEX NUMBERS AND MULTIPLE INTEGRALS

(2020 Admission Regular)

Time : 3 Hours Max. Marks : 80

SECTION – I

All the **first ten** questions are compulsory. They carry **1** mark each.

1. Find f_{xx} for the function $f(x,y) = 4x^3y - x^2y$.

2. Show that $2xy - 9x^2 + (2y + x^2 + 1)\frac{dy}{dx} = 0$ is exact.

3. Define divergence of a vector field.

4. Define del operator in Cartesian coordinates.

5. Write the value of $e^{3\pi i}$.

6. Find $\frac{dy}{dx}$ of $3\cosh(2x^4)$.

P.T.O.

7. Find $\frac{z_1}{z_2}$ where $z_1 = e^{3i}$ and $z_2 = e^{2i}$.

8. Evaluate $\int_{-3}^{2}\int_{0}^{1} y^2 x \, dy dx$.

9. Reverse the order of integration in $\int_{0}^{1}\int_{0}^{1-x} f(x,y) dy dx$.

10. Set up a double integral of $f(x, y)$ over the region given by $0 < y < 1, 0 < x < y^2$.

(10 × 1 = 10 Marks)

SECTION – II

Answer **any eight** questions from among the questions 11 to 26. These questions carry **2** marks each.

11. Find the total differential of the function $f(x, y) = \cos xy$.

12. Find $\frac{df}{dx}$ for the function $f(x,y) = x^2 + xy$, given that $y = \sin^{-1} x$.

13. Define Saddle point.

14. Find $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}$ for $\left(x^2 + y^2 + z^2\right)^{5/2} = 1$.

15. Find the Laplacian of the scalar field $\phi = x^2 yz$.

16. Find $curl \vec{F}$ for the vector field $\vec{F}(x,y,z) = yz\vec{i} + xy^2\vec{j} + yz^2\vec{k}$.

17. Find the direction in which the function $f(x,y) = xe^y$ increase interest at the point (2,0).

18. Find the gradient of the scalar field $\phi = x^2 y + yz$.

19. Write the real and imaginary part of $\frac{z}{z^*}$, where z^* is the conjugate of z.

20. Express $\sin 3\theta$ in terms of powers of $\sin\theta$.

21. Prove that $z^n - \frac{1}{z^n} = 2i \sin n\theta$, if $z = e^{i\theta}$.

22. Suppose $\sinh x = \frac{3}{4}$ find the exact value of x.

23. Evaluate the double integral $\iint_R y^2 x\, dA$ over the rectangle $R = \{(x, y); -3 \le x \le 2, 0 \le y \le 1\}$.

24. Evaluate $\int_0^4 \int_0^4 \int_0^4 kz\,dx\,dy\,dz$.

25. Find an expression for a volume element in spherical polar coordinates.

26. Find the Jacobian for $x = r \sin\theta \cos\phi,\ y = r \sin\theta \sin\phi, z = r \cos\theta$.

(8 × 2 = 16 Marks)

SECTION – III

Answer **any six** questions from among the questions 27 to 38. These questions carry 4 marks each.

27. Show that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ for the function $f(x, y) = y^3 \cos x$.

28. Find the Taylor expansion, up to quadratic terms in $x - 1$ and $y - 1$, of $2x^2 - 3xy + x$ about the point (1,1).

29. Discuss method of Lagrange undetermined multipliers.

30. A particle moves in three dimensional space with velocity $\vec{v}(t) = \vec{i} + t\vec{j} + t^2\vec{k}$ where t is the time variable. Find the position vector of the particle when $t = 1$ given that the particle is at $(-1,2,4)$ when $t = 0$.

31. If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ show that div grad $\left(\frac{1}{r}\right) = 0$.

32. Show that the acceleration of a particle travelling along a trajectory $r(t)$ is given by $a(t) = \frac{dv}{dt}\hat{t} + \frac{v^2}{\rho}\hat{n}$

where v is the speed of the particle, $\hat{t}$ is the unit tangent to the trajectory, $\hat{n}$ is its principal normal and ρ is its radius of curvature.

33. Prove that $\omega^3 = 1$ and $1 + \omega^2 + \omega^3 = 0$.

34. Solve the equation $z^4 - 3z^3 - 2z + 6 = 0$.

35. Show that i^i is a real number.

36. Draw the region of integration and evaluate $\int_0^{\pi/3}\int_0^{\cos y} x \sin y \, dx dy$.

37. Find the volume of the solid bounded by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 4$ and $z = 0$.

38. Evaluate $\int_0^8 \int_{y^{1/3}}^2 \sqrt{x^4 - 1} \, dx dy$.

(6 × 4 = 24 Marks)

SECTION – IV

Answer **any two** questions from among the questions 39 to 44. These questions carry **15** marks each.

39. (a) Locate all relative extrema and saddle points of

$$f(x,y) = 2x^3 + 6xy^2 - 3y^3 - 150x.$$

(b) The temperature of a point (x,y) on a unit circle is given by $T(x,y) = xy$. Find the temperature of the two hottest points on the circle.

40. (a) Find expressions For the equations of the tangent plane and line normal to the surface $\phi(x,y,z) = c$ at the point P with coordinates x_0, y_0, z_0. Use the results to Find the equations of the tangent plane and the line normal to the surface of the sphere $\phi = x^2 + y^2 + z^2 = a^2$ at the point $(0,0,a)$.

(b) Find $\nabla.(\nabla\times\vec{F})$ and $\nabla\times(\nabla\times\vec{F})$ where $\vec{F}(x,y,z) = \sin x\,\vec{i} + \cos(x-y)\,\vec{j} + z\vec{k}$.

41. (a) Find the derivative with respect to x of $e^{5x}(\cos 7x)$ using complex exponential.

(b) Evaluate the integral $I = \int e^{ax}\cos bx\,dx$.

42. (a) Use a triple integral to Find the volume of the solid within the cylinder $x^2 + y^2 = 9$ and between the planes $z = 1$ and $x + z = 5$.

(b) Evaluate $\iiint_G z\,dV$, where G is the wedge in the first octant cut off from the cylindrical solid $y^2 + z^2 \le 1$ and the planes $y = x$ and $x = 0$.

43. (a) If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$ find the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$.

(b) Show that $\nabla\times(\phi a) = \nabla\phi\times a + \phi\nabla\times a$.

44. (a) Solve the hyperbolic equation $\cosh x - 5\sinh x - 5 = 0$.

(b) Evaluate the double integral $I = \iint_R \left(a + \sqrt{x^2 + y^2}\right) dxdy$. where R is the region bounded by the circle $x^2 + y^2 = a^2$.

(2 × 15 = 30 Marks)

Section-I

All the first 10 questions are compulsory. They carry 1 mark each

1. $f_x = 12x^2y - 2xy;\ f_{xx} = 24xy - 2y$

2. $\frac{\partial}{\partial x}(2y + x^2 + 1) = 2x = \frac{\partial}{\partial y}(2xy - 9x^2)$

3. definition

4. $\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$

5. $e^{3\pi i} = \cos 3\pi + i\sin 3\pi = -1$

6. $\frac{dy}{dx} 3\cosh(2x^4) = 24x^3\sinh(2x^4)$

7. $z_1/z_2 = \frac{e^{3i}}{e^{2i}} = e^i = (\cos 1 + i\sin 1)$

8. $\int_{-3}^{2}\int_{0}^{1} y^2x\, dy dx = \int_{-3}^{2} \frac{x}{3}\, dx = -\frac{5}{6}$

9. $\int_0^1 \int_0^{1-y} f(x, y) dx dy$

10. $\int_{y=0}^{1} \int_{x=0}^{x=y^2} f(x, y) dx dy$

Section -II

Answer any 8 questions from among the questions 11 to 26. These questions carry 2 marks each

11. $df = f_x dx + f_y dy$ implies $df = -y\sin(xy)dx - x\sin(xy)dy$

12. $f_x = 2x + y, \quad f_y = x, \quad \frac{dy}{dx} = \frac{1}{(1-x^2)^{1/2}}$

$$\frac{df}{dx} = 2x + y + \frac{x}{(1-x^2)^{1/2}}$$

$$= 2x + \sin^{-1} x + \frac{x}{(1-x^2)^{1/2}}$$

13. A point at which a function of two variables has partial derivatives equal to zero but at which the function has neither a maximum nor a minimum value.

OR

a point on a curved surface at which the curvatures in two mutually perpendicular planes are of opposite signs

14. Differentiating partially w.r.t x

$$\frac{5}{2}(x^2+y^2+z^2)^{3/2}\left(2x+2z\frac{\partial z}{\partial x}\right)=0$$
$$\frac{\partial z}{\partial x}=-\frac{x}{z}$$

Differentiating partially w.r.t. y

$$\frac{5}{2}(x^2+y^2+z^2)^{3/2}\left(2y+2z\frac{\partial z}{\partial y}\right)=0$$
$$\frac{\partial z}{\partial y}=-\frac{y}{z}$$

$$\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=-\frac{x}{z}-\frac{y}{z}$$

15. $\frac{\partial^2}{\partial x^2}(x^2yz)+\frac{\partial^2}{\partial y^2}(x^2yz)+\frac{\partial^2}{\partial z^2}(x^2yz)=2yz$

16. $curl\,\vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ yz & xy^2 & yz^2\end{vmatrix}=z^2\vec{i}+y\vec{j}+(y^2-z)\vec{k}$

17. $\nabla f=e^y\mathrm{i}+xe^y\mathrm{j}$

The direction in which the function f increase fastest is $\nabla f(2,0)=\mathrm{i}+2\mathrm{j}$

18. $\text{grad}\,\phi=2xy\,\mathrm{i}+(x^2+z)\,\mathrm{j}+y\,\mathbf{k}$

19.

20. de Movire's theorem.

$$\begin{aligned}\cos 3\theta+i\sin 3\theta &=(\cos\theta+i\sin\theta)^3\\ &=(\cos^3\theta-3\cos\theta\sin^2\theta)+i(3\sin\theta\cos^2\theta-\sin^3\theta)\end{aligned}$$

$$\begin{aligned}\sin 3\theta &=3\sin\theta\cos^2\theta-\sin^3\theta\\ &=3\sin\theta-4\sin^3\theta\end{aligned}$$

21. $z=\cos\theta+i\sin\theta.$

$$z^n=(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta.$$
$$\frac{1}{z^n}=z^{-n}=(\cos\theta+i\sin\theta)^{-n}=\cos(-n\theta)+i\sin(-n\theta)$$
$$=\cos n\theta-i\sin n\theta$$
$$z^n-\frac{1}{z^n}=\cos n\theta+i\sin n\theta-\cos n\theta+i\sin n\theta$$
$$=2i\sin n\theta$$

22. $\sinh x = \frac{3}{4} \Rightarrow \frac{1}{2}(e^x - e^{-x}) = \frac{3}{4} \Rightarrow 2e^x - 3 - 2e^{-x} = 0$

multiplying by e^x

$$2e^{2x} - 3e^x - 2 = 0$$
$$(e^x - 2)(2e^x + 1) = 0$$
$$e^x = 2 \text{ or } e^x = -\frac{1}{2}$$

e^x is always positive $\Rightarrow e^x = 2 \Rightarrow x = \ln 2$

23.

$$\iint_R y^2x\, dA = \int_0^1 \int_{-3}^2 y^2 x\,dx\,dy = \int_0^1 \left[\frac{1}{2}y^2x^2\right]_{x=-3}^2 dy$$
$$= \int_0^1 \left(-\frac{5}{2}y^2\right)dy = -\frac{5}{6}y^3\Big|_0^1 = -\frac{5}{6}$$

24. $\int_0^4 \int_0^4 \int_0^4 kz\,dx\,dy\,dz = \int_0^4 \int_0^4 (kxz|_{x=0}^{x=4})\,dy\,dz$

$$= \int_0^4 \int_0^4 4kz\,dy\,dz = \int_0^4 (4kzy|_{y=0}^{y=4})\,dz$$
$$= \int_0^4 16kz\,dz = 8kz^2|_{z=0}^{z=4} = 128k$$

25. $dV = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} dr\,d\theta\,d\phi = r^2 \sin\theta\, dr\,d\theta\,d\phi$

26. Jacobian. $J = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ r\cos\theta\cos\phi & r\cos\theta\sin\phi & -r\sin\theta \\ -r\sin\theta\sin\phi & r\sin\theta\cos\phi & 0 \end{vmatrix}$

$$J = \cos\theta(r^2\sin\theta\cos\theta) + r\sin\theta(r\sin^2\theta)$$
$$= r^2\sin\theta(\cos^2\theta + \sin^2\theta) = r^2\sin\theta$$

Section -III

Answer any 6 questions from among the questions 27 to 38. These questions carry 4 marks each

27. $\frac{\partial}{\partial x}\left(y^3 \cos(x)\right) = -y^3 \sin(x)$

$$\frac{\partial^2}{\partial x \partial y}\left(y^3 \cos(x)\right) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\left(y^3 \cos(x)\right)\right) = \frac{\partial}{\partial y}(-y^3 \sin(x)) = -3y^2 \sin x$$

$$\frac{\partial}{\partial y}\left(y^3 \cos(x)\right) = 3y^2 \cos(x)$$

$$\frac{\partial^2}{\partial y \partial x}\left(y^3 \cos(x)\right) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(y^3 \cos(x)\right)\right) = \frac{\partial}{\partial x}(3y^2 \cos(x)) = -3y^2 \sin(x)$$

28. $fx = 4x - 3y + 1, \quad fx(1,1) = 2$

$fy = -3x, \quad fy(1,1) = -3$

$fxy = fyx = -3, \quad fxy(1,1) = -3$

$fxx = 4, \quad fxx(1,1) = 4$

$fyy = 0, \quad fyy(1,1) = 0$

Writing equation

$$f(x,y) \approx 0 + 2(x-1) - 3(y-1) + \frac{1}{2!}(4(x-1)^2 + 2(-3)(x-1)(y-1) + 0(y-1)^2)$$

29. To maximise f we require

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$$

If dx and dy were independent, we could conclude $f_x = 0 = f_y$. However, here they are not independent, but constrained because g is constant:

$$dg = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy = 0$$

Multiplying dg by an as yet unknown number λ and adding it to df

$$d(f + \lambda g) = \left(\frac{\partial f}{\partial x} + \lambda\frac{\partial g}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y} + \lambda\frac{\partial g}{\partial y}\right)dy = 0$$

where λ is called a Lagrange undetermined multiplier. In this equation dx and dy are to be independent and arbitrary: we must therefore choose λ such that

$$\frac{\partial f}{\partial x} + \lambda \frac{\partial g}{\partial x} = 0$$
$$\frac{\partial f}{\partial y} + \lambda \frac{\partial g}{\partial y} = 0$$

30. The position vector $\vec{r}(t)$ is

$$\vec{r}(t) = \int \vec{v}(t)dt = \int (\vec{i} + t\vec{j} + t^2\vec{k})dt = t\vec{i} + \frac{t^2}{2}\vec{j} + \frac{t^3}{3}\vec{k} + C$$

When $t = 0, \vec{r}(0) = -\vec{i} + 2\vec{j} + 4\vec{k}$

$\therefore C = -\vec{i} + 2\vec{j} + 4\vec{k}$

$\vec{r}(t) = t\vec{i} + \frac{t^2}{2}\vec{j} + \frac{t^3}{3}\vec{k} - \vec{i} + 2\vec{j} + 4\vec{k}$

When $t = 1, \vec{r}(1) = \frac{5}{2}\vec{j} + \frac{13}{3}\vec{k}$

The particle is at the point (0,5/2,13/3)

31.

$$\text{div grad}\left(\frac{1}{r}\right) = \nabla \cdot \nabla r^{-1} = \nabla \cdot ((-1)r^{-3}\vec{r})$$
$$= -[\nabla \cdot (r^{-3}r^{-1})] = -[r^{-3}(\nabla \cdot \vec{r}) + (\nabla r^{-3}) \cdot \vec{r}]$$
$$= -[r^{-3}(3) + (-3)r^{-5}\vec{r} \cdot \vec{r}]$$
$$= -[3r^{-3} - 3r^{-5}r^2] = 0$$

32.

The velocity of the particle is given by

$$\mathbf{v}(t) = \frac{dr}{dt} = \frac{dr}{ds}\frac{ds}{dt} = \frac{ds}{dt}\hat{\mathbf{t}}$$

where $v = ds/dt$ is the speed of the particle. Writing the velocity as $\mathbf{v} = v\hat{\mathbf{t}}$,

$$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \frac{dv}{dt}\hat{\mathbf{t}} + v\frac{d\hat{\mathbf{t}}}{dt}$$

$$\frac{d\hat{\mathbf{t}}}{dt} = \frac{ds}{dt}\frac{d\hat{\mathbf{t}}}{ds} = v\kappa\hat{\mathbf{n}} = \frac{v}{\rho}\hat{\mathbf{n}}$$

$$\mathbf{a}(t) = \frac{dv}{dt}\hat{\mathbf{t}} + \frac{v^2}{\rho}\hat{\mathbf{n}}$$

33.

$$\omega = -1/2 + i\sqrt{3}/2$$
$$\omega^2 = -1/2 - i\sqrt{3}/2$$
$$\omega^3 = \omega^2 \cdot \omega$$

$$= (-1/2 - i\sqrt{3}/2)(-1/2 + i\sqrt{3}/2)$$
$$= (-1/2)^2 - (i\sqrt{3}/2)^2$$
$$= 1/4 + 3/4 = 1$$

$$1 + \omega^2 + \omega^3 = 1 + (-1/2 + i\sqrt{3}/2) + (-1/2 - i\sqrt{3}/2) = 0$$

34. $z = 3$ is a solution

$$z^4 - 3z^3 - 2z + 6 = (z - 3)(z^3 - 2)$$

$$z^3 \quad = 2 \cdot 1 = 2 \cdot e^{i2k\pi}$$

$$z = 2^{1/3} e^{i2k\pi/3}, k = 0,1,2$$

$$k = 0, \quad z = 2^{1/3} e^0 = 2^{1/3}$$
$$k = 1, z = 2^{1/3} e^{i2\pi/3} = 2^{\frac{1}{3}}(-\frac{1}{2} + \frac{\sqrt{3}}{2} i)$$
$$k = 2, z = 2^{1/3} e^{i4\pi/3} = 2^{\frac{1}{3}}(-\frac{1}{2} - \frac{\sqrt{3}}{2} i)$$

Solutions are

$$z_1 = 3$$
$$z_2 = 2^{1/3}$$
$$z_3 = 2^{\frac{1}{3}}(-\frac{1}{2} + \frac{\sqrt{3}i}{2})$$
$$z_4 = 2^{1/3}\left(-\frac{1}{2} - \frac{\sqrt{3}i}{2}\right)$$

35. Let $z = i^i$

Taking *ln* on both sides,

$$ln(z) = i\, ln(i)$$

$$i = e^{i(\frac{\pi}{2} + 2n\pi)}$$
$$\ln i = i\left(\frac{\pi}{2} + 2n\pi\right)$$

$$\ln z = (i)(i)\left(\frac{\pi}{2} + 2n\pi\right)$$
$$\ln z = -\frac{\pi}{2} - 2n\pi$$

Taking exponential on both sides, $z = i^i = e^{-\pi/2-2n\pi}$

Principle value at $n = 0 \Rightarrow (i)^i = e^{-\pi/2}$ is a real numbes.

36. drawing region

$$\int_0^{\pi/3}\int_0^{\cos y} x\sin y\,dx\,dy = \int_0^{\pi/3}\left[\int_0^{\cos y} x\sin y\,dx\right]dy = \int_0^{\pi/3} \frac{x^2}{2}\sin y\Big]_{x=0}^{\cos y} dy$$

$$= \int_0^{\pi/3}\left[\frac{1}{2}\cos^2 y\sin y\right]dy = -\frac{1}{6}\cos^3 y\Big]_0^{\pi/3} = \frac{7}{48}$$

37. The solid is bounded above by the plane $z = 4 - y$ and below by the region R within the circle $x^2 + y^2 = 4$.

Volume, $V = \iint_R (4 - y)dA$

$$V = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4 - y)dydx = \int_{-2}^{2}\left[4y - \frac{1}{2}y^2\right]_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$= \int_{-2}^{2} 8\sqrt{4 - x^2}dx = 8(2\pi) = 16\pi$$

38. Reverse the order of integration

$0 \le x \le 2$

$0 \le y \le x^3$

$$\int_0^8\int_{\sqrt[3]{y}}^2 \sqrt{x^4 + 1}dxdy = \int_0^2\int_0^{x^3} \sqrt{x^4 + 1}dydx$$

$$= \int_0^2 y\sqrt{x^4 + 1}\Big|_0^{x^3} dx$$

$$= \int_0^2 x^3\sqrt{x^4 + 1}dx = \frac{1}{6}\left(17^{\frac{3}{2}} - 1\right)$$

Section -IV

Answer any 2 questions from among the questions 39 to 44. These questions carry 15 marks each

39. a)

$$f_x = 6x^2 + 6y^2 - 150$$
$$f_y = 12xy - 9y^2$$
$$f_{xx} = 12x$$
$$f_{yy} = 12x - 18y$$
$$f_{xy} = 12y$$

For stationary points,

$$6x^2 + 6y^2 - 150 = 0 \text{ and } 12xy - 9y^2 = 0$$

$$x^2 + y^2 = 25 \text{ and } y(4x - 3y) = 0$$

If $y = 0 \Rightarrow x^2 = 25 \Rightarrow x = \pm 5$ giving $(5,0)$ and $(-5,0)$

If $4x = 3y$ then $x = \frac{3}{4}y \Rightarrow \frac{9}{16}y^2 + y^2 = 25 \Rightarrow y = \pm 4.$

$y = 4 \Rightarrow x = 3$ and $y = -4$ gives $x = -3 \Rightarrow (3,4)$ and $(-3,-4)$

there are four stationary points $(5,0)$, $(-5,0)$, $(3,4)$ and $(-3,-4)$.

stationary point $(5,0)$, $f_{xx}f_{yy} - f_{xy}^2 = 60^2 > 0$, $f_{xx} = 60 > 0$ and $f_{yy} = 60 > 0$.

Hence $(5,0)$ is a minimum.

For the stationary point $(-5,0)$, $f_{xx}f_{yy} - f_{xy}^2 = (-60)^2 > 0$, $f_{xx} = -60 < 0$ and $f_{yy} = -60 < 0$. Hence $(-5,0)$ is a maximum.

For the stationary point $(3,4$, $f_{xx}f_{yy} - f_{xy}^2 = -3600 < 0$ so $(3,4)$ is a saddle.

For the stationary point$(-3,-4)$, $f_{xx}f_{yy} - f_{xy}^2 = -3600 < 0$ so $(-3,-4)$ is a saddle.

39.b)

maximise $T(x,y)$ subject to the constraint $x^2 + y^2 = 1$.

$$y + 2\lambda x = 0$$
$$x + 2\lambda y = 0$$

These results, together with the original constraint $x^2 + y^2 = 1$.

Solving $\Rightarrow \lambda = \pm 1/2$, implies that $y = \mp x$.

$$y = x \Rightarrow x = \pm\frac{1}{\sqrt{2}},\ y = \pm\frac{1}{\sqrt{2}}$$
$$y = -x \Rightarrow x = \mp\frac{1}{\sqrt{2}},\ y = \pm\frac{1}{\sqrt{2}}$$

substitute the four pairs of x - and y - values into $T(x, y) = xy$

maximum temperature on the unit circle is $T_{max} = 1/2$ at the points $y = x = \pm 1/\sqrt{2}$

40. a)

A vector normal to the surface $\phi(x, y, z) = c$ at the point P is simply $\nabla\phi$ evaluated at that point; we denote it by $\mathbf{n}_0$. If $\mathbf{r}_0$ is the position vector of the point P relative to the origin, and $\mathbf{r}$ is the position vector of any point on the tangent plane, then the vector equation of the tangent plane is,

$(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n}_0 = 0$

Similarly, if $\mathbf{r}$ is the position vector of any point on the straight line passing through P (with position vector $\mathbf{r}_0$) in the direction of the normal $\mathbf{n}_0$ then the vector equation of this line is

$(\mathbf{r} - \mathbf{r}_0) \times \mathbf{n}_0 = 0$

For the surface of the sphere $\phi = x^2 + y^2 + z^2 = a^2$

$$\begin{aligned}\nabla\phi &= 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}\\ &= 2a\mathbf{k} \text{ at the point } (0,0,a)\end{aligned}$$

equation of the tangent plane to the sphere at this point is

$(\mathbf{r} - \mathbf{r}_0) \cdot 2a\mathbf{k} = 0$

This gives $2a(z - a) = 0$ or $z = a$, as expected. The equation of the line normal to the sphere at the point$(0,0,a)$ is

$(\mathbf{r} - \mathbf{r}_0) \times 2a\mathbf{k} = 0$

which gives $2ay\mathbf{i} - 2ax\mathbf{j} = 0$ or $x = y = 0$, i.e. the z-axis, as expected. The tangent plane and normal to the surface of the sphere at this point.

40. b) $\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin x & \cos(x-y) & z \end{vmatrix} = -\sin(x-y)\,\vec{k}$

$$\nabla \cdot (\nabla \times \vec{F}) = \frac{\partial}{\partial z}(-\sin(x-y)) = 0$$

$$\nabla \times (\nabla \times \vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & -\sin(x-y) \end{vmatrix}$$

$$= \cos(x-y)\,\vec{i} + \cos(x-y)\,\vec{j}$$

41. a) $z = e^{5x}(\cos 7x + i\sin 7x) = e^{5x}e^{7ix} = e^{(5+7i)x}$

This complex number has $e^{5x}\cos 7x$ as its real part.

differentiating z with respect to x

$$\frac{dz}{dx} = (5+7i)e^{(5+7i)x} = (5+7i)e^{5x}(\cos 7x + i\sin 7x)$$

Equating real parts $\Rightarrow \frac{d}{dx}(e^{8x}\sin 2x) = e^{5x}(5\cos 7x - 7\sin 7x)$

41) b)

Let us consider the integrand as the real part of the complex number

$$e^{ax}(\cos bx + i\sin bx) = e^{ax}e^{ibx} = e^{(a+ib)x}$$

$$\begin{aligned} \int e^{(a+ib)x}dx &= \frac{e^{(a+ib)x}}{a+ib} + c \\ &= \frac{(a-ib)e^{(a+ib)x}}{(a-ib)(a+ib)} + c \\ &= \frac{e^{ax}}{a^2+b^2}(ae^{ibx} - ibe^{ibx}) + c \end{aligned}$$

where $c = c_1 + ic_2$ and equating real parts

$$I = \int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + c_1$$

42. a) The lower surface of the solid is the plane $z = 1$ and the upper surface is the plane $x + z = 5$ or, equivalently, $z = 5 - x$.

$$\text{volume of } G = \iiint_G dV = \iint_R \left[\int_1^{5-x} dz\right] dA$$

$$\text{volume of } G = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_1^{5-x} dzdydx = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} z\Big|_{z=1}^{5-x} dydx$$

$$= \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (4-x)dydx = \int_{-3}^{3} (8-2x)\sqrt{9-x^2}dx$$

$$= 8\int_{-3}^{3} \sqrt{9-x^2}dx - \int_{-3}^{3} 2x\sqrt{9-x^2}dx$$

$$= 8\left(\frac{9}{2}\pi\right) - \int_{-3}^{3} 2x\sqrt{9-x^2}dx$$

$$= 8\left(\frac{9}{2}\pi\right) - 0 = 36\pi$$

42.b) $\int_0^1 \int_0^y \int_0^{\sqrt{1-y^2}} zdzdxdy$

$\int_0^1 (y - y^3)dy = \frac{1}{8}$

43. a)

$$r = \frac{x}{y}, s = \frac{y}{z}, t = \frac{z}{x} \dots\dots(1)$$

$$\frac{\partial u}{\partial x} = \frac{1}{y}\frac{\partial u}{\partial r} - \frac{z}{x^2}\frac{\partial u}{\partial t} \dots\dots(1)$$

$$\frac{\partial u}{\partial y} = \frac{1}{z}\frac{\partial u}{\partial r} - \frac{x}{y^2}\frac{\partial u}{\partial r} \dots\dots(1)$$

$$\frac{\partial u}{\partial z} = \frac{1}{x}\frac{\partial u}{\partial t} - \frac{y}{z^2}\frac{\partial u}{\partial s} \dots\dots(1)$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0 \dots\dots\dots(3)$$

43. b) The x-component of the LHS is

$$\frac{\partial}{\partial y}(\phi a_z) - \frac{\partial}{\partial z}(\phi a_y) = \phi\frac{\partial a_z}{\partial y} + \frac{\partial \phi}{\partial y}a_z - \phi\frac{\partial a_y}{\partial z} - \frac{\partial \phi}{\partial z}a_y$$

$$= \phi\left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right) + \left(\frac{\partial \phi}{\partial y}a_z - \frac{\partial \phi}{\partial z}a_y\right)$$

$$= \phi(\nabla \times \mathbf{a})_x + (\nabla\phi \times \mathbf{a})_x$$

y and z components can be similarly found.

44.a)

$$\frac{e^x + e^{-x}}{2} - 5\left(\frac{e^x - e^{-x}}{2}\right) - 5 = 0$$

$$\frac{-2e^{2x} + 3}{e^x} - 5 = 0$$

$$-2e^{2x} + 3 - 5e^x = 0$$

$$e^x = -3, \frac{1}{2}$$

$$x = \ln(-3) \text{ or } x = -\ln(2)$$

44. b)

$$I = \int_{-a}^{a} dx \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy\left(a + \sqrt{x^2 + y^2}\right)$$

$x = \rho\cos\phi$ and $y = \rho\sin\phi$

$$I = \iint_{R'} (a + \rho)\left|\frac{\partial(x, y)}{\partial(\rho, \phi)}\right| d\rho d\phi$$

where R' is the rectangular region in the $\rho\phi$-plane whose sides are $\rho = 0, \rho = a, \phi = 0$ and $\phi = 2\pi$.

$$J = \frac{\partial(x, y)}{\partial(\rho, \phi)} = \begin{vmatrix} \cos\phi & \sin\phi \\ -\rho\sin\phi & \rho\cos\phi \end{vmatrix} = \rho(\cos^2\phi + \sin^2\phi) = \rho$$

$$dxdy = \rho d\rho d\phi$$

Therefore.

$$I = \iint_{R'} (a + \rho)\rho d\rho d\phi$$

$$= \int_0^{2\pi} d\phi \int_0^a d\rho(a + \rho)\rho = 2\pi\left[\frac{a\rho^2}{2} + \frac{\rho^3}{3}\right]_0^a = \frac{5\pi a^3}{3}$$