(Pages : 3) **S – 6789**

Reg. No. : ....

Name : ....

Third Semester M.Sc. Degree Examination, February 2024

Mathematics

MM 232 : FUNCTIONAL ANALYSIS – I

(2020 Admission onwards)

Time : 3 Hours Max. Marks : 75

PART – A

Answer any **five** questions. **Each** question carries **3** marks.

1. Prove that $l_{p'}$ is a subspace of l_p for every $1 \leq p' \leq p \leq \infty$.

2. Define linear functional on a normed linear space. Give an example of a discontinuous linear functional.

3. Let X be a normed linear space and $\{X_1, ..., x_n\}$ be a linearly independent set in X. Prove that there exist $x_1', ..., x_n' \in X'$ such that $x_j'(x_i) = \begin{cases} 1 \text{ if } i = j; \\ 0 \text{ if } i \neq j \end{cases}$.

4. Define Schaunder basis of a normed linear space.

5. State and prove Banach-Steinhaus theorem.

6. State Gauss formulae.

7. Let X and Y be normed linear spaces and $F \in BL(X,Y)$. Show that F'' extends F linearly, preserving its norm.

8. Show that a weak convergent sequence in a normed linear space has a unique weak limit.

(5 × 3 = 15 Marks)

P.T.O.

PART – B

Answer **all** questions. **Each** question carries **12** marks.

9. (A) (a) If E is a convex subset of a normed linear space X, prove that E° and $\overline{E}$ are also convex in X. **4**

(b) Let X be a normed linear space and Y be a subspace of X. Prove that $Y \neq X$ if and only if $Y^\circ = \phi$. **3**

(c) Let X be a normed linear space over K and Y be a subspace of X of finite dimension n. Prove that every bijective linear map $F : K^* \to Y$ is also a homeomorphism. **5**

OR

(B) (a) Prove that $X = l_p$ is a normed linear space for every p satisfying $1 \leq p \leq \infty$. **5**

(b) Le X and Y be two normed linear spaces over K. Prove that $BL(X,Y)$ is a linear space over K. **7**

10. (A) (a) Let X be a normed linear space, E an open convex subset of X and Y a subspace of X such that $E \cap Y = \phi$. If Y is not a hyperspace in X, prove that there exists $x \in x$ such that $x \notin Y$ and $E \cap$ span $\{Y, x\} = \phi$. **7**

(b) Let X be a normed linear space and Y a closed subspace of X. Prove that Y with induced norm and X/Y with quotient norm are Banach if and only if X is Banach. **5**

OR

(B) (a) State and prove the Hahn-Banach separation theorem. **6**

(b) Show that a normed linear space X is a Banach space if and only if every absolutely summable series of elements in X is summable in X. **6**

11. (A) (a) State and prove the Uniform boundedness principle. **6**

(b) Let X be Banach space over $\mathbb{C}$, D an open subset of $\mathbb{C}$ and $F : D \to X$. Prove that F is analytic on D if and only if $x' \circ F$ is analytic on D for every $x' \in X'$. **6**

OR

(B) State and prove Open Mapping Theorem. **12**

12. (A) (a) Let X be a Banach space. Show that $BL(X)$ is a Banach algebra. **6**

(b) Let X be one of the Banach sequence spaces $l_p (1 \le p \le \infty)$, with appropriate norm. Let $A(x(1), x(2), ...,)) = (x(1), x(2)/2, x(3)/3, ...,))$. Prove that $s(A) = a(A) = e(A) \cup \{0\}$. **6**

OR

(B) (a) Let $A \in BL(X)$. If E is a set of eigenvectors of A such that no two elements of E correspond to the same eigen value of A, prove that E is linearly independent in X. **6**

(b) State and prove Spectral radius formulae for $A \in BL(X)$ where X is a Banach space over $\mathbb{C}$. **6**

13. (A) (a) State and prove a criterion for weak convergence of a sequence in $C([a,b])$. **6**

(b) Let X be a normed linear space, S a finite dimensional subspace of X', and x'' an element of X''. Prove that for every $\varepsilon > 0$, there exists $x \in X$ such that $\|x\| < \|x''\| + \varepsilon$. **6**

OR

(B) (a) Let X be a reflexive normed linear space. Prove that every closed subspace of X and X' are reflexive. **6**

(b) Let X and Y be normed linear spaces and $F : X \to Y$ be linear. If F is compact, show that it sends every weak convergent in X to a convergent sequence in Y. Further prove that the converse holds if X is reflexive. **6**

(5 × 12 = 60 Marks)
