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Reg. No. : ....

Name : ....

Second Semester M.Sc. Degree Examination, November 2023

Mathematics

MM 222 : REAL ANALYSIS II

(2020 – Admission onwards)

Time : 3 Hours Max. Marks : 75

PART – A

Answer any **five** questions. **Each** question carries **3** marks.

1. Prove that Lebesgue outer measure is translation invariant.
2. State true or false; Justify your assertion. A set is countable if and only if its measure is zero.
3. Show that $\int_1^\infty \frac{dx}{x} = \infty$.
4. Let f be defined on [0,1] by $f(x)=0$ if $x \in \mathbb{Q}$ and $f(x)=1$ if $x \notin \mathbb{Q}$. Find the four derivatives at any x.
5. Show that if μ is a non-negative set function on a ring is countably additive and is finite on some set, then μ is a measure.
6. Let $f = g\,a.e.(\mu)$ where μ is a complete measure. Show that if f is measurable then g is measurable.
7. Prove that if $f, g \in L^p(\mu)$ and a, b are constants then $af + bg \in L^p(\mu)$.
8. Let $v(E) = \int_E xe^{-x^2} dx$. What are the positive, negative and null sets with respect to v?

(5 $\times$ 3 = 15 Marks)

P.T.O.

PART – B

Answer **all** questions choosing either (A) or (B). **Each** question carries **12** marks.

9. (A) (a) Prove that the collection of measurable sets is a $\sigma -$ algebra. **6**

(b) Prove that the Lebesgue outer measure of an interval is its length. **6**

OR

(B) (a) Let E be a given set. Prove that the following five statements are equivalent:

(i) E is measurable

(ii) Given $\in > 0$, there is an open set $O \supset E$ with $m^*(O - E) < \in$.

(iii) Given $\in > 0$, there is a closed set $F \supset E$ with $m^*(E - F) < \in$.

(iv) There is a G in G_δ with $E \subset G$, $m^*(G - E) = 0$.

(v) There is an F in F_σ with $F \subset E$, $m^*(E - F) = 0$. **12**

10. (A) State and prove Fatou's lemma. Hence state and prove Lebesgue's Monotone convergence theorem. **12**

OR

(B) *Let* $[a, b]$ be a finite interval and let $f \in L(a, b)$ with indefinite integral F. Prove that $F' = f$ a.e. in $[a, b]$. **12**

11. (A) Prove that if μ is a $\sigma -$ finite measure on a ring $\mathbb{R}$, that it has a unique extension to the $\sigma -$ ring $\mathbb{S}(\mathbb{R})$. **12**

OR

(B) If μ is a measure on a $\sigma -$ ring $\mathbb{S}$, then prove that the class $\overline{\mathbb{S}}$. of sets of the form $E \Delta N$ for any sets E, N such that $E \in \mathbb{S}$ while N is contained in some set in $\mathbb{S}$ of zero measure, is a $\sigma -$ ring and the set function $\overline{\mu}$ defined by $\overline{\mu}(E \Delta N) = \mu(E)$ is a complete measure on $\overline{\mathbb{S}}$. **12**

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12. (A) State and prove Holder's inequality. Also discuss when equality occurs in case when f and g are non-negative measurable functions. **12**

OR

(B) Prove that for $p \geq 1, L^p(\mu)$ is a complete metric space. **12**

13. (A) Prove that if f_n is a sequence of measurable functions which is fundamental in measure, then there exists a measurable function f such that $f_n \to f$ in measure. **12**

OR

(B) State and prove that Radon-Nikodym theorem. **12**

(5 $\times$ 12 = 60 Marks)
